Quiz 3 Solution

November 4, 2018

Use gaussian elimination to solve the following two problems involving the three vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \text{ and } \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^4$.

1. Find all vectors $\vec{w}$ in $\mathbb{R}^4$ that are orthogonal to $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

2. Determine the values for c such that

$$\begin{bmatrix} 1 \\ c \\ 1 \\ c \end{bmatrix} \in \text{Span}\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}.$$

Solution to Problem 1. $\vec{w}$ is orthogonal to $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ if and only if

$$\vec{v}_1 \cdot \vec{w} = 0, \ \vec{v}_2 \cdot \vec{w} = 0, \text{ and } \vec{v}_3 \cdot \vec{w} = 0,$$

or equivalently, the components of $\vec{w}$ comprise a solution to the system

$$\begin{aligned} w_1 + w_2 + w_3 + 2w_4 &= 0 \\ w_1 + w_2 + 2w_3 + w_4 &= 0 \\ w_1 + 2w_2 + w_3 + w_4 &= 0. \end{aligned} \tag{1}$$

To solve this system we reduce the augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 \end{bmatrix},$$

by performing the following elementary row operations.

Step 1. Subtract the first row from the second row to obtain

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & 2 & 1 & 1 & 0 \end{bmatrix}.$$

Step 2. Subtract the first row from the third row to obtain

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix}.$$

Step 3. Swap the second and third rows to obtain

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}.$$

Step 4. Subtract the second row from the first row to obtain

$$\begin{bmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}.$$

Step 5. Subtract the third row from the first row to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}.$$

Thus, w_4 is a nonpivotal variable and the general solution to (1) is given by choosing w_4 to be an arbitrary real number and then defining w_1, w_2, and w_3 so that

$$\begin{aligned} w_1 \qquad\quad + 4w_4 &= 0 \\ w_2 \qquad - w_4 &= 0 \\ w_3 - w_4 &= 0. \end{aligned}$$

Therefore the general vector $\vec{w}$ orthogonal to $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ has the form

$$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} -4w_4 \\ w_4 \\ w_4 \\ w_4 \end{bmatrix} = w_4 \begin{bmatrix} -4 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

where w_4 is an arbitrary real number.

Solution to Problem 2.

$$\begin{bmatrix} 1 \\ c \\ 1 \\ c \end{bmatrix} \in \text{Span}\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\} \tag{2}$$

if and only if there exist real numbers x_1, x_2, x_3 such that

$$\begin{bmatrix} 1 \\ c \\ 1 \\ c \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix},$$

or equivalently, the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + x_2 + 2x_3 &= c \\ x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= c \end{aligned} \tag{3}$$

is solvable. The augmented matrix associated with this system is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & c \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & c. \end{bmatrix}$$

which we reduce via the following row operations.

Step 1. Subtract the first row from the second and third rows and subtract 2 times the first row from the fourth row to obtain

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & c-1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & c-2 \end{bmatrix}.$$

Step 2. Swap the second and third rows to obtain

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c-1 \\ 0 & -1 & -1 & c-2 \end{bmatrix}.$$

Step 3. Subtract the second row from the first row and add the second row to the fourth row to obtain

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c-1 \\ 0 & 0 & -1 & c-2 \end{bmatrix}.$$

Step 4. Subtract the third row from the first row and add the third row to the fourth row to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 2-c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c-1 \\ 0 & 0 & 0 & 2c-3 \end{bmatrix}.$$

Now, the system (3) is solvable if and only if this last matrix does not have a pivot in the fourth row, i.e., $2c - 3 = 0$. Therefore, (2) holds if and only if $c = 3/2$.